

A Note on Repeated Games with Non-Monotonic Value

LEEAT YARIV¹

School of Mathematical Sciences, Raymond and Beverly Sackler, Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv 69978 Tel Aviv, Israel

Abstract: We show that even when the information structure is independent of the state of nature, the value of the n -stage zero-sum game with incomplete information is not necessarily monotonic with respect to the length of the game. More precisely, we give an example of such an n -stage game in which $V_1 > V_2 < V_3$.

Key Words: Repeated games with incomplete information, value, imperfect monitoring.

1 Introduction

Suppose K is a finite set of states of nature, each corresponding to a payoff matrix of a zero-sum game. Two players, PI and PII, take part in the n -stage game, denoted G_n , played as follows: nature chooses a state k from the set K according to a probability distribution P . PI is informed about the state chosen. PII has only the knowledge of P . Then, the players repeat n times the zero-sum game defined by the payoff matrix A_k .

If the information is standard, that is, after each stage all past actions of both players are announced, then the value V_n of the above described game is monotonic (decreasing) with respect to n . This is due to a recursive structure inherent to the sequence V_n , $n \geq 1$. The intuitive reasoning for this is that the information PII receives from PI increases with the number of stages played. This information is utilized by PII to improve his average payoff (see Sorin [1980]).

Suppose now that the game is played with imperfect monitoring. There exist two information functions, L_1 and L_2 such that after each stage in which player i played action a_i , player j gets the information signal $L_j(a_1, a_2)$. Note that games with perfect monitoring are a special case of games with imperfect monitoring taking the information functions to be one-to-one on the set of ordered pairs of actions.

In general games with imperfect monitoring, a recursive structure fails to exist since PI might be unable to estimate precisely the information gathered

¹ I am very grateful to Ehud Lehrer who introduced this question to me.

by PII during the game. Therefore, the proof of monotonicity of V_n is no longer valid.

It has been shown by Lehrer [1987] that if the information functions are dependent on the state of nature, then the sequence V_n may be non-monotonic. The example Lehrer gives hinges heavily on the fact that the information is state-dependent. This note is devoted to the case of state-independent information structure. Therefore, the signals received during the game imply only the actions taken by both players and cannot reveal directly the state of nature. We show that V_n may be non-monotonic also when restricting the signalling to be state-independent.

2 The Example

Let $S = (s_i)_{i=1}^6$ and $T = (t_j)_{j=1}^5$ be the possible actions of players PI and PII, respectively, in the game where the state space is $K = \{1, 2, 3\}$ and the distribution over K is $P = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$.

Let the payoff matrices be:

$$A_1 = \begin{bmatrix} 40 & 4 & 0 & 4 & 4 \\ -100 & -100 & -100 & -100 & -100 \\ -100 & -100 & -100 & -100 & -100 \\ 40 & 0 & 4 & 4 & 4 \\ -100 & -100 & -100 & -100 & -100 \\ -100 & -100 & -100 & -100 & -100 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -100 & -100 & -100 & -100 & -100 \\ 0 & 40 & 4 & 4 & 4 \\ -100 & -100 & -100 & -100 & -100 \\ -100 & -100 & -100 & -100 & -100 \\ 4 & 40 & 0 & 4 & 4 \\ -100 & -100 & -100 & -100 & -100 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -100 & -100 & -100 & -100 & -100 \\ -100 & -100 & -100 & -100 & -100 \\ 4 & 0 & 40 & 4 & 4 \\ -100 & -100 & -100 & -100 & -100 \\ -100 & -100 & -100 & -100 & -100 \\ 0 & 4 & 40 & 4 & 4 \end{bmatrix}$$

Suppose that the information matrices are:

$$B_1 = \begin{bmatrix} 1 & 2 & 3 & N & N \\ 1 & 2 & 3 & N & N \\ 1 & 2 & 3 & N & N \\ 1 & 2 & 3 & N & N \\ 1 & 2 & 3 & N & N \\ 1 & 2 & 3 & N & N \end{bmatrix} \quad B_2 = \begin{bmatrix} N & N & N & 12 & 31 \\ N & N & N & 12 & 32 \\ N & N & N & 23 & 31 \\ N & N & N & 13 & 21 \\ N & N & N & 23 & 21 \\ N & N & N & 13 & 32 \end{bmatrix}$$

In other words, when PI plays action s_j and PII plays action t_m player i receives the signal $L_i(s_j, t_m) = B_i(j, m)$ for all $i \in \{1, 2\}$ $j \in \{1, \dots, 6\}$ $m \in \{1, \dots, 5\}$.

Note that in each information matrix, there is an informative part (containing the signals 12, 13, 23, 21, 23, 32, 1, 2, 3) and a non-informative part (in which the signal is null: N).

It is assumed that at each stage the players know, in addition to the signals announced, the actions they themselves chose in all previous stages.

The intuitive description of the proof is as follows. For each state $k \in K$, very low payoffs appear in four rows of the payoff matrix A_k . This forces PI to use, throughout games in which the number of stages is small, one out of two actions (depending on the state k). PII, knowing this, can deduce after the first stage of any short enough game a new partition of the space of states. Namely, he can “cross out” one of the states and consider only two out of the three states as being possibly played. Moreover, the information structure allows PII to do so without enabling PI to evaluate exactly which state has been “crossed out” by PII. An action of PII which realizes the information he has gathered contributes no additional information to PII. However, such an action discloses completely his information to PI, who can play optimally against PII from then, on, and hence increase the payoffs.

Specifically, in the 2-stage game PII, realizing the information he has gathered during the first stage of the game, can decrease the payoff in the second stage and hence $V_1 > V_2$.

In the 3-stage game, if PII uses the information he has gathered on the second stage, his private knowledge is revealed to PI and the payoff on the third stage will increase. Otherwise, during stage 2, PII will have to ignore the information he has already gathered. This leads to the inequality $V_2 < V_3$.

Note that such a construction can not be made when the number of possible states of nature is two. “Crossing out” of a matrix in that case is equivalent to PII knowing exactly which state has been chosen (having a degenerate partition of the space of states).

3 Formal Proof

Denote by G_n the n -fold repeated game.

In G_1 , if PII plays his pure strategy t_4 , he guarantees a payoff ≤ 4 . If PI plays his

pure strategy s_k when informed that state k has been chosen, he ensures himself a payoff ≥ 4 . Hence, $V_1 = 4$.

In G_2 , PII can ensure a payoff ≤ 3 by using the following strategy:

Stage 1: play t_4 or t_5 with an equal probability of $\frac{1}{2}$.

Stage 2: play t_1 if 23 or 32 have been signalled.

play t_2 if 13 or 31 have been signalled.

play t_3 if 12 or 21 have been signalled.

The logical interpretation of this strategy is that the differences between the payoffs appearing in the game matrices make deceiving for PI (playing as if the state of nature were different from the state k actually chosen, i.e. not playing s_k or s_{k+3}) unworthy. Thus, if the signal $10*i + j$, $1 \leq i, j \leq 3$, has been announced to PII after the first stage of the game, the state of nature is very likely to be i or j .

Mathematically, we may assume, from symmetry considerations, that $k = 1$. If at the first stage PI has chosen s_2, s_3, s_5 , or s_6 , the lump payoff doesn't exceed $\frac{-100 + 40}{2} = -30 < 3$. Otherwise, again from symmetry considerations, we may

assume s_1 was played at the first stage. Thus PII got the signal 12 or 31. PI does not know which of these signals PII got, but he knows PII got each of these signals with probability $\frac{1}{2}$. Therefore, PI can do no better than to expect PII to play t_2 or t_3 with probability $\frac{1}{2}$ each and to ensure an expected payoff of 2 in the second stage. Hence, $V_2 \leq \max\left(-30, \frac{(4+2)}{2}\right) = 3$.

In G_3 , PI can ensure himself a payoff $\geq \frac{10}{3}$ by playing the following strategy, assuming k is the state of nature that has been chosen:

Stages 1, 2: play each one of s_k, s_{k+3} with probability $\frac{1}{2}$.

Stage 3: If N was signalled in the second stage play s_k, s_{k+3} with probability $\frac{1}{2}$ each. Otherwise, play the following:

$k = 1$: $(\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0)$ or s_1 or s_4 for 1 or 2 or 3 respectively;

$k = 2$: $(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0)$ or s_2 or s_5 for 2 or 3 or 1 respectively;

$k = 3$: $(0, 0, \frac{1}{2}, 0, 0, \frac{1}{2})$ or s_3 or s_6 for 3 or 1 or 2 respectively;

(the notation (q_1, \dots, q_6) , $\sum_{i=1}^6 q_i = 1$, is used to denote the one-stage mixed action of PI where the action s_i is played with probability q_i , $1 \leq i \leq 6$)

It suffices to show that PI, using the above strategy, gets payoff of at least $\frac{10}{3}$ against any pure strategy of PII. Let $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ be a pure strategy of PII. σ_i is a function from PII's histories of length $i - 1$ stages to the space of actions available for him.

We denote $\varphi(23) = \varphi(32) = t_1$, $\varphi(13) = \varphi(31) = t_2$, and $\varphi(12) = \varphi(21) = t_3$. θ_i symbolizes the signal received by PII after stage i .

If $\sigma_1 \in \{t_1, t_2, t_3\}$ then PI gets payoff of 40 with probability $\frac{1}{3}$ and so the expected 3-stage payoff is $\geq \frac{\frac{1}{3} * 40}{3} \geq \frac{10}{3}$.

Thus, we may assume $\sigma_1 \in \{t_4, t_5\}$, guaranteeing a payoff of 4 and a signal $\theta_1 \neq N$ on the first stage.

If $\sigma_2(\theta_1) \in \{t_4, t_5\}$, then the payoff in the second stage is 4 and, since PI ensures himself a payoff of 2 on the third stage, the payoff in G_3 is $\geq \frac{4 + 4 + 2}{3} = \frac{10}{3}$.

Suppose now that $\sigma_2(\theta_1) \in \{t_1, t_2, t_3\}$ and $\sigma_2(\theta_1) \neq \varphi(\theta_1)$, then with probability $\frac{1}{2}$, $\sigma_2(\theta_1) = t_k$, where k is the state of nature that has been chosen, and the payoff in the second stage of the game is 40. Thus the expected payoff in G_3 is at least $\frac{4 + \frac{1}{2} * 40}{3} \geq \frac{10}{3}$.

If $\sigma_2(\theta_1) = \varphi(\theta_1)$, PII ensures a payoff of 2 in the second stage, but no information is augmented for him ($\theta_2 = N$). Moreover, all the information PII has is revealed. PI is, therefore, able to play optimally on the third stage according to the following two cases:

1. If $\sigma_3(\theta_1, N) \in \{t_1, t_2, t_3\}$ and $\sigma_3(\theta_1, N) \neq \sigma_2(\theta_1)$, then the payoff in the third stage is 40 with probability $\frac{1}{2}$ and hence gives expected total payoff of at least $\frac{4 + 2 + \frac{1}{2} * 40}{3} = \frac{26}{3} > \frac{10}{3}$ in G_3 .
2. If $\sigma_3(\theta_1, N) \in \{\sigma_2(\theta_1), t_4, t_5\}$, then the above strategy of PI clearly ensures a payoff of 4 on the third stage and so guarantees a payoff of $\frac{(4 + 2 + 4)}{3} = \frac{10}{3}$ in G_3 .

Finally, we deduce $V_3 \geq \frac{10}{3} > 3 \geq V_2 < 4 = V_1$ and the sequence V_n is non-monotonic.

4 Comments

4.1 It is easily verified that restricting the information signals to be common to both players (as well as independent of the state of nature) does not guarantee monotonicity of the value. For instance, suppose we add to each *payoff* matrix in the game described above four columns on the right identical to the right most column in the original matrix. Furthermore, suppose the information matrix is:

$$B = \begin{bmatrix} 1 & 2 & 3 & a & a & b & b & c & c \\ 1 & 2 & 3 & a & b & b & c & c & a \\ 1 & 2 & 3 & b & b & c & c & a & a \\ 1 & 2 & 3 & b & c & c & a & a & b \\ 1 & 2 & 3 & c & c & a & a & b & b \\ 1 & 2 & 3 & c & a & a & b & b & c \end{bmatrix}$$

Then we get a game with a sequence of values (V_n) identical to the one obtained in the original example. In particular, the values are not monotonic.

4.2 Similar considerations may be used to construct a game with observable payoffs (i.e., the signals, for both players at each stage, being the payoffs achieved) where the corresponding sequence (V_n) is non-monotonic. For example, consider the game with state space $K = \{1, 2, 3\}$, distribution over K : $P = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, and payoff matrices:

$$A_1 = \begin{bmatrix} h & 4 & -4 & 1 & 1 & 2 & 2 & 3 & 3 \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ h & -4 & 4 & 2 & 3 & 3 & 1 & 1 & 2 \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -4 & h & 4 & 2 & 2 & 3 & 3 & 1 & 1 \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ 4 & h & -4 & 3 & 1 & 1 & 2 & 2 & 3 \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ 4 & -4 & h & 3 & 3 & 1 & 1 & 2 & 2 \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -4 & 4 & h & 1 & 2 & 2 & 3 & 3 & 1 \end{bmatrix}$$

where $h = 100$.

Analogous calculations to those introduced in this note lead to the conclusion that for this game, $V_1 = 2 > V_2 = 1 < V_3 = \frac{4}{3}$.

References

- Sorin S (1980) An introduction to two-person zero-sum repeated games with incomplete information. Technical report no 312 IMSS – Economics, Stanford University, Stanford, CA
 Lehrer E (1987) A note on the monotonicity of V_n . *Economics Letters* 23: 341–342

Received: March 1995

Revised Version: August 1995