Online Appendix for "Who Cares More? Allocation with Diverse Preference Intensities"

Preliminary Results

Lemma A1 *Agents of type P are strictly more risk averse than agents of type I for lotteries whose support includes $\diamond$.*

Proof of Lemma A1. Denote $k_j(x) = u_j''(x)/u_j'(x)$ for any $j \in \{P, I\}$ and $x \in (0,\infty)$. Recall that $u_j'(x) < 0$, and $u_j''(\cdot)$, $u_j'(\cdot)$ are continuous functions, thus $k_j(\cdot)$ is well-defined. Consider an ordinary differential equation (ODE) $v'(x) = k_j(x)\cdot v(x)$ with initial condition $v(x_0) = u_j'(x_0)$; it has a unique solution for $x \in (0,\infty)$ given by $v(x) = u_j'(x) = u_j'(x_0)\cdot \exp[\int_{x_0}^{x} k_j(z)dz]$. Similarly, an ODE $w'(x) = v(x)$ with boundary condition $w(x_0) = u_j(x_0)$ has a unique solution equal to $u_j(x)$, providing the representation

$$u_j(x) \;=\; u_j'(x_0)\cdot \int_{x_0}^{x} \exp\left[\int_{x_0}^{y} k_j(z)dz\right]dy + u_j(x_0). \tag{5}$$

Since $\lim_{x\to\infty} u_j(x) = 0$ for $j \in \{P, I\}$, $u_j'(x) < 0$, and $k_P(x) > k_I(x)$ for all $x \in (0,\infty)$, then

$$\frac{u_P(x)}{-u_P'(x)} \;=\; \int_x^{\infty} \exp\left[\int_x^{y} k_P(z)dz\right]dy \;>\; \int_x^{\infty} \exp\left[\int_x^{y} k_I(z)dz\right]dy \;=\; \frac{u_I(x)}{-u_I'(x)}, \tag{6}$$

for any $x \in (0,\infty)$. Consider lottery $q_\lambda = \lambda\delta_{x_1} + (1-\lambda)\delta_\diamond$ for arbitrary $x_1 \in (0,\infty)$. Let $\lambda_j(x_1, x_2) \in (0,1)$ be such that an agent of type j is indifferent between $q_{\lambda_j(x_1,x_2)}$ and the degenerate lottery δ_{x_2}. Then,

$$\lambda_j \;=\; \frac{u_j(x_2)}{u_j(x_1)} \;=\; 1 - \frac{\int_{x_1}^{x2} \exp\left[\int_{x_1}^{y} k_j(z)dz\right]dy}{\int_{x_1}^{\infty} \exp\left[\int_{x_1}^{y} k_j(z)dz\right]dy} \;=\; 1 - \frac{1}{1 + a_j(x_1, x_2)},$$

where

$$a_j(x_1, x_2) = \left(\int_{x_1}^{x_2} \exp\left[-\int_{y}^{x_2} k_j(z)dz\right]dy\right)^{-1} \cdot \int_{x_2}^{\infty} \exp\left[\int_{x_2}^{y} k_j(z)dz\right]dy.$$

Since $k_P(z) \geq k_I(z)$, then

$$\int_{x_1}^{x_2} \exp\left[-\int_{y}^{x_2} k_P(z)dz\right]dy < \int_{x_1}^{x_2} \exp\left[-\int_{y}^{x_2} k_I(z)dz\right]dy,$$

and

$$\int_{x_2}^{\infty} \exp\left[\int_{x_2}^{y} k_P(z)dz\right]dy > \int_{x_2}^{\infty} \exp\left[\int_{x_2}^{y} k_I(z)dz\right]dy.$$

Therefore, $a_P(x_1, a_2) > a_I(x_1, x_2)$, and $\lambda_P(x_1, x_2) > \lambda_I(x_1, x_2)$ for all $0 < x_1 < x_2 < \infty$. Similarly, consider $\lambda_j(0, x_2) = \frac{u_j(x_2)}{u_j(0)}$. Since $u_j(x)$ is continuous at $x = 0$, then $\lambda_P(x, 0.5x_2) > \lambda_I(x, 0.5x_2)$ for $x \longrightarrow +0$ implies $\lambda_P(0, 0.5x_2) \geq \lambda_I(0, 0.5x_2)$. Thus, $\lambda_P(0, x_2) = \lambda_P(0, 0.5x_2) \cdot \lambda_P(0.5x_2, x_2) > \lambda_I(0, 0.5x_2) \cdot \lambda_I(0.5x_2, x_2) = \lambda_P(0, x_2)$.

Consider now an arbitrary lottery q with support on $[0, X] \cup \{\diamond\}$ such that $q(\diamond) > 0$. Let x_1 be a certainty equivalent of $q \mid [0, X]$ for agent of type I. Then, $V_I(q) = (1 - q(\diamond)) \cdot V_I(q \mid [0, X]) = (1 - q(\diamond)) \cdot u_I(x_1)$. Since a P-agent is more risk averse than an I-agent for lotteries with support on $[0, X]$, we have $V_P(q) = (1 - q(\diamond)) \cdot V_P(q \mid [0, X]) \leq (1 - q(\diamond)) \cdot u_P(x_1)$. Let x_2 be the certainty equivalent of $(1 - q(\diamond)) \cdot \delta_{x_1} + q(\diamond) \cdot \delta_\diamond$ for agent I; clearly, such a finite x_2 exists, since u_I is strictly decreasing, and $\lim_{x \to \infty} u_I(x) = u_I(\diamond) = 0$. Then, $1 - q(\diamond) = \lambda_I(x_1, x_2) = \frac{u_I(x_2)}{u_I(x_1)} < \frac{u_P(x_2)}{u_P(x_1)}$. We conclude that $V_P(\delta_{x_2}) = u_P(x_2) > (1 - q(\diamond)) u_P(x_1) \geq V_P(q)$, while $V_I(\delta_{x_2}) = V_I(q)$. Thus, the certainty equivalent of an arbitrary non-degenerate lottery q for an I-agent is strictly lower utility-wise, than the certainty equivalent of q for a P-agent. We conclude that P-agent is strictly more risk averse, than I-agent. ■

Lemma A2 $\gamma(x_1) = \dfrac{u_P(x_1) - u_P(x_2(x_1))}{u_I(x_1) - u_I(x_2(x_1))}$ *is strictly increasing.*

Proof of Lemma A2. Denoting by γ' the derivative of γ, we have

$$\operatorname{sign}(\gamma') = \operatorname{sign}\Big(\big(u'_P(x_1) - u'_P(x_2) \cdot x'_2\big)\big(u_I(x_1) - u_I(x_2)\big) - \big(u'_I(x_1) - u'_I(x_2) \cdot x'_2\big)\big(u_P(x_1) - u_P(x_2)\big)\Big) =$$

$$= \operatorname{sign}\Big(u'_P(x_1) \cdot \Big(1 - x'_2 \cdot \exp\Big[\int_{x_1}^{x_2} k_P(z)dz\Big]\Big) \cdot (-u'_I(x_1)) \cdot \int_{x_1}^{x_2} \exp\Big[\int_{x_1}^{y} k_I(z)dz\Big] dy \; -$$

$$- \; u'_I(x_1) \cdot \Big(1 - x'_2 \cdot \exp\Big[\int_{x_1}^{x_2} k_I(z)dz\Big]\Big) \cdot (-u'_P(x_1)) \cdot \int_{x_1}^{x_2} \exp\Big[\int_{x_1}^{y} k_P(z)dz\Big] dy\Big) \; =$$

$$= \operatorname{sign}\Big(\int_{x_1}^{x_2} \Big(\exp\Big[\int_{x_1}^{y} k_P(z)dz\Big] - \exp\Big[\int_{x_1}^{y} k_I(z)dz\Big]\Big) dy \; + \; x'_2 \cdot \exp\Big[\int_{x_1}^{x_2} k_P(z)dz\Big] \cdot$$

$$\cdot \exp\Big[\int_{x_1}^{x_2} k_I(z)dz\Big] \cdot \Big(\int_{x_1}^{x_2} \exp\Big[-\int_{y}^{x_2} k_I(z)dz\Big] dy - \int_{x_1}^{x_2} \exp\Big[-\int_{y}^{x_2} k_P(z)dz\Big] dy\Big)\Big) \; > \; 0$$

where we used $u'_I(x_1), u'_P(x_1) < 0$, $x'_2 = f(x_1)/f(x_2) > 0$, and $k_P(x) > k_I(x)$ for all $x \in (0, X)$. ■

Lemma A3 *A fair competitive equilibrium exists.*

Proof of Lemma A3. Consider a class of allocations, parameterized by $y = (y_2, ..., y_N) \in Y \equiv [0,1]^{N-1}$ as follows:

$$q_k(y) = f \,\Big|\, \Big[x_k(y), x_{k-1}(y)\Big] \cup \Big[x^{k-1}(y), x^k(y)\Big] \text{ for } k = 2, ..., N \quad , \quad q_1(y) = f \,\Big|\, \Big[x_1(y), x^1(y)\Big],$$

where

$$x_k(y) \;=\; F^{-1}\Big(\sum_{i=k+1}^{N} y_k \mu_k\Big) \quad , \quad x^k(y) \;=\; F^{-1}\Big(\sum_{i=1}^{k-1} \mu_i + \sum_{i=k}^{N} y_k \mu_k\Big).$$

Thus, each $q_k(y)$ for $k = 2, ..., N$ consists of two blocks: one associated with higher-quality goods $\big[x_k(y), x_{k-1}(y)\big]$, one associated with lower-quality goods $\big[x^{k-1}(y), x^k(y)\big]$. The value of y_k encodes the probability k-agents get a good in the first, higher-quality block.

Define the functions $\alpha_k, b_k : Y \longrightarrow \mathbb{R}$ and $v_k : [0, \overline{X}] \times Y \longrightarrow \mathbb{R}$ for $k = 1, 2, ..., N$ recursively as follows. First, set $\alpha_N \equiv 1$, $b_N \equiv 0$, $v_N(x, y) \equiv u_N(x)$ for $x \in [0, \overline{X}]$. For any $k < N$, if $\alpha_N, ..., \alpha_{k+1}$, $b_N, ..., b_{k+1}$, v_{k+1} have been defined, set

$$\alpha_k(y) \;=\; \frac{v_{k+1}(x_k, y) - v_{k+1}(x^k, y)}{u_k(x_k) - u_k(x^k)} \quad , \quad b_k(y) = v_{k+1}(x_k, y) - \alpha_k(y) u_k(x_k) \quad , \quad \text{and}$$

$$v_k(x, y) \;=\; \max\{\, \alpha_N(y) u_N(x) + b_N(y) \,,\, \dots \,,\, \alpha_k(y) u_k(x) + b_k(y) \,\},$$

where the dependence of x_k and x^k on y is implicit.

Let $p(x, y) \equiv v_1(x, y)$ and Further define $\omega_k : Y \longrightarrow \mathbb{R}$ for $k = 1, ..., N$ as follows:

$$\omega_k(y) \;=\; \int_0^{\overline{X}} p(x, y) q_k(y)(x) dx.$$

That is, $\omega_k(y)$ captures the k-agents' expenditure given the price schedule $p(x, y)$. Define $I : Y \longrightarrow \mathbb{R}$ by

$$I(y) \;=\; \left(\sum_{i=1}^{N} \mu_i\right)^{-1} \cdot \int_0^{\overline{X}} p(x, y) f(x) dx.$$

The value of $I(y)$ captures the effective per-person income in the economy. Finally, define $\phi : Y \longrightarrow Y$ by

$$\phi(y)_k \;=\; y_k + \frac{2}{\pi} \cdot \Big[(1 - y_k) \cdot \mathbb{1}\{I \geq \omega_k\} + y_k \cdot \mathbb{1}\{I \leq \omega_k\}\Big] \cdot \arctan(I - w_k) \quad , \quad k = 2, ..., N.$$

The function ϕ offers one way to continuously map an unbounded domain to a compact interval.

It is straightforward to see that x_k, x^k, α_k, b_k, v_k, p, w_k, I, and ϕ are continuous functions of y. By Brouwer's Fixed Point Theorem, since Y is a compact convex set, ϕ has a fixed point $y^* \in Y$. We now show that $(q(y^*), p(y^*))$ is a fair competitive equilibrium. In what follows, for simplicity, we suppress the dependence on y whenever this dependence is clear.

Claim G1. For all $y \in Y$, for all $k = N, N-1, ..., 1$, $\alpha_k > 0$, $v_k(\cdot, y)$ is a strictly decreasing function, $v_k(x, y) > 0$ for all $x \in [0, \overline{X}]$, and $b_k \leq 0$.

Proof. We prove the statement by induction on $k = N, N-1, ..., 1$. By definition, $\alpha_N = 1$, $b_N = 0$, and $v_N(x) = u(x)$. Thus, the statement holds for $k = N$. Assume the statement holds for $N, ..., k+1$, and consider α_k, $v_k(x)$, and b_k. Since $u_k(\cdot)$ and $v_{k+1}(\cdot)$ are strictly decreasing, then $\alpha_k > 0$. Since $\alpha_N, ..., \alpha_k > 0$, it follows that $v_k(\cdot)$ is the maximum of a finite number of strictly decreasing functions and, hence, strictly decreasing. Furthermore, $v_k(x) \geq \alpha_N u_N(x) + b_N = u_N(x) > 0$. Finally, consider

$$b_k \;=\; v_{k+1}(x_k) - \alpha_k u_k(x_k) \;=\; -\;\frac{v_{k+1}(x_k)u_k(x^k)}{u_k(x_k) - u_k(x^k)} \cdot \left(1 - \frac{v_{k+1}(x^k)}{v_{k+1}(x_k)} \cdot \frac{u_k(x_k)}{u_k(x^k)}\right)$$

Thus, to prove that $b_k \leq 0$, it suffices to show that $\frac{v_{k+1}(x^k)}{v_{k+1}(x_k)} \leq \frac{u_k(x^k)}{u_k(x_k)}$. By the definition of v_{k+1}, there is $j > k$ such that $v_{k+1}(x^k) = \alpha_j u_j(x^k) + b_j$. Since $\alpha_j > 0$, $v_j(x) > 0$, and $b_j \leq 0$ by the induction assumption, and $u_j(x^k) < u_j(x_k)$, we have

$$\frac{v_{k+1}(x^k)}{v_{k+1}(x_k)} \;=\; \frac{\alpha_j u_j(x^k) + b_j}{v_{k+1}(x_k)} \;\leq\; \frac{\alpha_j u_j(x^k) + b_j}{\alpha_j u_j(x_k) + b_j} \;\leq\; \frac{u_j(x^k)}{u_j(x_k)}.$$

It suffices to show that $\frac{u_j(x^k)}{u_j(x_k)} \leq \frac{u_k(x^k)}{u_k(x_k)}$ for $j > k$. From eq. (6) in the proof of Lemma A1 above, applied to $I = j$ and $P = k$, we get

$$\psi_j(x) \;\equiv\; \ln(u_j(x))' \;<\; \ln(u_k(x))' = \psi_k(x)$$

for all x. This implies that

$$\frac{u_j(x^k)}{u_j(x_k)} \;=\; \exp\left(\int_{x_k}^{x^k} \psi_j(x)dx\right) \;<\; \exp\left(\int_{x_k}^{x^k} \psi_k(x)dx\right) \;=\; \frac{u_k(x^k)}{u_k(x_k)},$$

proving the claim. □

Claim G1 implies that $p(x,y) > 0$. Clearly, $p(\cdot, y)$ is a measurable function, and so $p(y^*) = p(\cdot, y^*)$ is a valid price schedule. The market-clearing condition holds for $(q(y^*), p(y^*))$ by construction.

We now show that $\omega_k = I > 0$ for all k. Denote by $J_+ = \{k \in \{1, 2, ..., N\} \mid \omega_k > I\}$, and $J_- = \{k \in \{1, 2, ..., N\} \mid \omega_k < I\}$. Consider an arbitrary $k > 1$. Since y^* is a fixed point of ϕ, then $\omega_k > I$ implies

$y_k^* = 0$, and $\omega_k < I$ implies $y_k^* = 1$. Assume, towards a contradiction, that $k \in J_+$. Then, $y_k^* = 0$ and, since the price schedule is strictly decreasing, $j < k$ implies $j \in J_+$. Since I is a weighted average of ω_k, then $I_+ \neq \emptyset$ implies $I_- \neq \emptyset$. Thus, there is a type $i > k > 1$ such that $i \in I_-$. Then, $q_i = f \mid [x_i, x_{i-1}]$ and $q_k = f \mid [x^{k-1}, x^k]$. It follows that $\omega_i > \omega_k > I > \omega_i$, in contradiction. Similarly, assume towards a contradiction, that $k \in J_-$, then $y_k^* = 1$ and, since price schedule is strictly decreasing, $j < k$ implies $j \in J_-$. Thus, there is type $i > k > 1$ such that $i \in J_+$ and we get $\omega_k > \omega_i > I > \omega_k$, in contradiction. We conclude that $\omega_k = I$ for all $k = 2, ..., N$. Therefore, $\omega_1 = I$ as well. Finally, $I > 0$ since $p(x, y^*) > 0$.

It remains to show that q_k solves the consumer's problem for each type k, given price p and endowment $\omega_k = I$. The argument above shows that $y_k^* \in (0,1)$ for all $k = 2, ..., N$. Indeed, if $y_k^* = 0$, then $I = \omega_k < \omega_1 = I$; If $y_k^* = 1$, then $I = \omega_k > \omega_1 = I$. Denote by $x_0 = x^0 = (x_1 + x^1)/2$.

Claim G2. For all k, if $x \in [x_k, x_{k-1}] \cup [x^{k-1}, x^k]$ then $\alpha_k u_k(x) + b_k = p(x)$.

Proof. The proof mimics the proof of Claim B3 in the proof of Proposition 4 and Corollary 4 in the main text, where Case 1 applies for all k, and $p(x) = v_1(x)$. □

Assume that $V_k(q') > V_k(q_k)$ for some feasible measure q', then by Claim G1 and Claim G2:[29]

$$\int_0^{\overline{X}} p(x)q'(x)dx \;\geq\; \int_0^{\overline{X}} (\alpha_k u_k(x) + b_k)q'(x)dx \;=\; \alpha_k V_k(q') + b_k \cdot q'\Big([0, \overline{X}]\Big) \;\geq\; \alpha_k V_k(q') + b_k \;>$$

$$> \; \alpha_k V_k(q) + b_k \;=\; \int_0^{\overline{X}} (\alpha_k u_k(x) + b_k)q_k(x)dx \;=\; \int_0^{\overline{X}} p(x)q_k(x)dx = \omega_k$$

Thus, q' violates the budget constraint for type-k agents. We conclude that q_k is an optimal allocation for those agents. ■

Restrictions on Allocations

In this section, we first consider a relaxation of our setting, whereby the mechanism designer can reduce goods' quality. We show such an option would never be utilized in the second-best solution. We then turn to settings in which all agents need to be served with certainty. The

[29] We abuse notation denoting by $\int h(x)q'(x)dx$ the integral of the function h with respect to measure q', although q' may not have a density function.

second-best solution inherits the qualitative features of the solution identified in the main text. The proofs of results presented in this section are relegated to the end of our discussion.

Damaged Goods

In many settings, the mechanism designer can lower the quality of available goods: appointments can be delayed, vacant units of public-housing can be assigned at future times. In fact, whenever considering similar goods that differ in their delivery times, the possibility of damage is closely linked to storage opportunities.

Allowing for artificial reduction of quality relaxes the feasibility constraint. For any allocation (q_p, q_I), denote by Q_P, Q_I the cumulative distributions on $[0, X] \cup \{\diamond\}$. The possibility of damage then changes the feasibility constraint to

$$\mu_P Q_P(x) + \mu_I Q_I(x) \leq F(x) \quad \forall x \in [0, X].$$

We refer to the corresponding social planner and mechanism designer's problems as the *relaxed* problems. Their respective solutions are then the *relaxed first-best* and *relaxed second-best*.

For the social planner, damaging goods cannot be beneficial. As it turns out, it is not useful for the mechanism designer either. Indeed, it is never useful to provide P-agents damaged goods for the same reasoning underlying the lack of gaps in their service (see Lemma 3). Similarly, it is never useful to provide I-agents damaged goods following arguments akin to those justifying the possibility of disposal (see Lemma 4). Thus, we have the following:

Proposition 7. *The relaxed first-best and second-best solutions coincide with the first-best and second-best solutions, respectively.*

Restricting Disposal

Our analysis assumes that the mechanism designer has the option to leave some agents without any good even when there is sufficient supply. In some applications, however, denial of service may not be acceptable: for example, leaving families without public housing while some apartments sit empty.

Suppose the mechanism designer faces the additional constraint that all agents be served with a good of quality in $[0, X]$. That is, $q_I([0, X]) = q_P([0, X]) = 1$. We call the corresponding problem the *restricted mechanism designer problem*, its solution the *restricted second-best*.

As before, in the restricted second-best, no type can receive an allocation that dominates that of other types. Arguments similar to those above also imply that the restricted second-best cannot exhibit an inverted spread. However, with no disposal available, Lemma 4 may not hold, and I-agents may see a gap in the support of their allocation; instead of receiving no good at all, they now receive goods of the lowest quality.

Proposition 8. *There exists a unique solution of the restricted mechanism designer's problem, given by*

$$q_P = f \Big| [x_1, x_2]$$

$$q_I = f \Big| [0, x_1] \cup [x_2, x_3] \cup [x_4, X],$$

where $0 < x_1 < x_2 \leq x_3 \leq x_4 \leq X$, $F(x_2) - F(x_1) = \mu_P$, *and* $F(x_1) + (F(x_3) - F(x_2)) + (F(X) - F(x_4)) = \mu_I$. *Furthermore, whenever the restricted second-best solution exhibits a gap,* $x_3 < x_4 < X$, *the second-best solution exhibits disposal.*

The restricted second-best allocation has a "modified" IPI structure. All P-agents are still served in a contiguous block in between I-agents, but I agents may experience a gap in the quality of goods they receive.

Welfare What does the no-disposal restriction imply on welfare? In the unrestricted problem, disposal was used only when IC_{PI} binds: in all other cases, the solution is identical and so is the resulting welfare.

When the second-best solution admits disposal, Corollary 3 implies that I-agents strictly prefer the second-best to the pooling allocation, while P-agents strictly prefer the pooling to the second-best. Consider the polar case in which the mechanism designer cannot offer goods of quality lower than $\overline{X}$. P-agents must remain indifferent between their allocation and I-agents' allocation, and therefore any mixture of the two; without disposal, however, the pooling allocation is a mixture of the two allocations, which implies that P agents must be indifferent between their allocation and the pooling allocation. It follows that P-agents are made strictly better off by the ban on disposal. Since overall welfare must be reduced by the ban on disposal, I-agents are made strictly worse off. As it turns out, this intuition carries over even when the mechanism designer can offer goods of quality bounded by $X > \overline{X}$.

Corollary 6. *Suppose the second-best solution admits disposal. Then, I-agents strictly prefer the second-best solution to the restricted second-best solution, while P-agents strictly prefer the restricted second-best solution to the second-best solution.*

Thus, policies designed to protect I-agents by ensuring they all receive goods may, in fact, decrease their welfare. At the same time, such policies guarantee that no agent remains without a good, thus reducing welfare heterogeneity within I-agents.

Proofs Pertaining to Restrictions on Allocations

Proof of Proposition 7. We say that an allocation q exhibits storage if it satisfies the relaxed feasibility condition $\mu_P Q_P(x) + \mu_I Q_I(x) \leq F(x)$, but violates the feasibility condition of the original problem. That is, if $(f - \mu_P q_P - \mu_I q_I)(A) < 0$ for some $A \subseteq [0, X]$. To prove the proposition, it suffices to show that the relaxed first-best and second-best solutions never exhibit storage.

By construction, storage implies that some goods are lowered in quality before being served, implying that higher-quality, feasible goods are unused. Therefore, if an allocation exhibits storage, it exhibits disposal. Since both the relaxed and the original first-best solutions never exhibit disposal, they never exhibit storage either.

Consider a relaxed second-best allocation q. The allocation q does not exhibit an inverted spread, following the same argument used in the original problem. Indeed, Lemma 2* and its proof do not rely on $q(\cdot)$ being non-atomic, and can thus be replicated. Furthermore, if IC_{jk} is not binding, q does not exhibit disposal for k-type agents. Otherwise, we could increase the mass of k-type agents served by a sufficiently small amount so that IC_{jk} is preserved, generating an incentive-compatible allocation producing higher welfare.

If neither IC constraint binds, the relaxed second-best coincides with the relaxed first-best that, as we have already established, does not exhibit storage.

If both IC constraints bind, both agents are indifferent between allocations q_P, q_I and $\dfrac{\mu_P q_P + \mu_I q_I}{\mu_P + \mu_I}$. If q exhibits storage, it must exhibit disposal, implying that the pooling allocation $q_I^{pool} = q_P^{pool} = f \mid [0, \overline{X}]$ must be strictly preferred by both agents over $\dfrac{\mu_P q_P + \mu_I q_I}{\mu_P + \mu_I}$ and, hence, strictly preferred to q_P and q_I. Since the pooling allocation is also incentive compatible, q cannot be the relaxed second-best solution, a contradiction.

If IC_{PI} binds and IC_{IP} does not, we have seen that q does not exhibit disposal for P-agents.

Since q also does not exhibit inverted spread, then $q_P = f \mid [x_1, x_2]$ for some $0 < x_1 < x_2 < \overline{X}$ and $q_I(x) = \mu_I^{-1} \cdot f(x)$ for all $x \in [0, x_1]$; Otherwise, there would be disposal for P-types. Thus, any storage must take place for goods of quality below x_2. Assume now, towards a contradiction, that q exhibits storage. Then, we must have $(f - \mu_P q_P - \mu_I q_I)(A) > 0$ and $(f - \mu_I q_I)(B) < 0$ for some $[0, x_2] \triangleleft A \triangleleft B \triangleleft \{\diamond\}$. By continuity, there exists $\gamma \in (0,1)$ and a positive-measure set $C \subset B$ such that P-agents are indifferent between $q_I|C$ and $\gamma(f - \mu_P q_P - \mu_I q_I) \mid A + (1-\gamma)\delta_\diamond$. But then, I-agents must strictly prefer the latter. We can thus replace the allocation of q_I in C with this lottery, strictly improving welfare but maintaining incentive compatibility. This implies that q is not a relaxed second-best, in contradiction.

Finally, consider the case in which IC_{IP} binds and IC_{PI} does not. Let $x_1 = \inf\{\text{supp}(q_P) \cap [0, X]\}$ and $x_2 = \sup\{\text{supp}(q_P) \cap [0, X]\}$ (both of which are well defined since $q_P \neq \delta_\diamond$). Since IC_{PI} does not bind, as we have argued above, I-agents' allocation does not exhibit disposal. Define $x' = \sup\{\text{supp}(q_I)\}$. Since there is no disposal for I-agents, we must have $q_I([0, x']) = 1$ and $f(x) = \mu_P q_P(x) + \mu_I q_I(x)$ for $x \in [0, x']$. If $x' \in (x_1, x_2]$, then q exhibits an inverted spread, which cannot occur in a relaxed second-best. If $x' \leq x_1$, then q_I dominates q_P, which also cannot occur in a relaxed second-best. Therefore, we must have $x' \geq x_2$, implying that P-agents' allocation does not exhibit disposal. This means that q does not exhibit disposal, and thus does not exhibit storage. ■

Proof of Proposition 8.

The proof of the existence of the restricted second-best solution mimics the proof of the existence of the second-best solution (Proposition 0) with the only difference that we restrict attention to the closed subset of distributions such that $q_i\big([0, X]\big) = 1$.

If the second-best solution does not exhibit disposal, then the restricted second-best solution coincides with the second-best solution and the statement holds. The second-best solution never exhibits a gap. Thus, if the restricted second-best solution exhibits a gap, the restricted and unrestricted solutions differ and the second-best solution must exhibit disposal.

Suppose the second-best solution exhibits disposal. We show that the restricted second-best solution $q = (q_P, q_I)$ has the asserted structure. It is straightforward to see that Lemmas 2 and 3, as well as Corollary 2 from the main text continue to hold for the restricted second-best solution as well. In addition, Claim A1 used in the proof of Lemma 4 in the Appendix of the main text continues to hold for the restricted second-best solution. It follows that $q_P = f \mid [x_1, x_2]$, and for any $x \in [0, x_1)$, $q_I(x) = \mu_I^{-1} f(x)$.

Claim H1. There are no sets $A, B, C \subseteq [0, X]$ such that $A \triangleleft B \triangleleft C$, $q_I(B) > 0$, $\big(f - \mu_P q_P - \mu_I q_I\big)(A) > 0$, and $\big(f - \mu_P q_P - \mu_I q_I\big)(C) > 0$.

Proof. Assume, towards a contradiction, that such sets A, B,and C exist. Let $\gamma \in (0,1)$ be such that P-agents are indifferent between lotteries $\gamma \cdot (f - \mu_P q_P - \mu_I q_I) \mid A + (1-\gamma) \cdot (f - \mu_P q_P - \mu_I q_I) \mid C$ and $q_I \mid B$. I-agents then strictly prefer the first of these two lotteries. For small enough $\epsilon > 0$, the following allocation is feasible:

$$q_I' = q_I + \epsilon \cdot \Big[\gamma \cdot (f - \mu_P q_P - \mu_I q_I) \mid A + (1-\gamma) \cdot (f - \mu_P q_P - \mu_I q_I) \mid C - q_I \mid B\Big] \quad , \quad q_P' = q_P$$

Moreover, $V_P(q_P') = V_P(q_P) \geq V_P(q_I) = V_P(q_I')$, and $V_I(q_I') > V_I(q_I) \geq V_I(q_P) = V_I(q_P')$. Thus, q is not a restricted second-best solution, in contradiction. □

Denote $x_3 = \inf\big(\mathrm{supp}(f - \mu_P q_P - \mu_I q_I) \cap [x_2, X]\big)$ and $x_4 = \sup\big(\mathrm{supp}(f - \mu_P q_P - \mu_I q_I) \cap [x_2, X]\big)$. By Claim H1, $q_I([x_3, x_4]) = 0$. It follows that $q_I = f \mid [0, x_1] \cup [x_2, x_3] \cup [x_4, X]$.

Finally, suppose q' is another restricted second-best solution. Then $q_P' = f \mid [x_1', x_2']$, and $q_I' = f \mid [0, x_1'] \cup [x_2', x_3'] \cup [x_4', X]$ for some x_i', $i = 1,2,3,4$. Since the optimized function and the constraints are convex, then $q'' = 0.5q + 0.5q'$ is also a restricted second-best solution. It should be that $q_P'' = f \mid [x_1'', x_2'']$, and $q_I'' = f \mid [0, x_1''] \cup [x_2'', x_3''] \cup [x_4'', X]$ for some x_i'', $i = 1,2,3,4$, which is possible only if $x_i = x_i' = x_i''$ for $i = 1,2,3,4$. Hence, $q'' = q' = q$, proving the uniqueness of the restricted second-best solution. ■

Proof of Corollary 6.

Define the augmented supply function $\hat{f}$ on $[0, \infty)$ by $\hat{f}(x) = f(x)$ for $x \in [0, X]$ and $\hat{f}(x) = f(X)$ for $x > X$. Consider a family of supply functions $f_{\hat{X}}$ with $\hat{X} \in [X, \infty)$ defined by $f_{\hat{X}} = \hat{f}(x)$ for $x \in [0, \hat{X}]$ and $f_{\hat{X}}(x) = 0$ for $x > \hat{X}$. By Proposition 3, the second-best allocation is identical for all supply functions $f_{\hat{X}}$. However, the restricted second-best allocation may depend on $\hat{X}$. By Proposition 8, $q_P = f \mid [x_1, x_2]$ for the restricted second-best allocation. We first show that x_1 is an increasing function of $\hat{X}$, which implies that $V_P(q_P)$ is a decreasing function of $\hat{X}$. We then consider a limit $\hat{X} \longrightarrow \infty$ to compare the constrained second-best allocation to the second-best allocation.

The first-best allocation for any supply $f_{\hat{X}}$ does not depend on the value of $\hat{X}$. We denote it by q^{FB}.

If IC_{PI} is not violated for q^{FB}, then either the second-best allocation coincides with q^{FB}, or IC_{IP} is not binding for the second-best allocation. In both cases, the second-best allocation does not exhibit disposal, and Corollary 6 holds vacuously.

Suppose IC_{IP} is violated for q^{FB}, in which case IC_{PI} does not bind for q^{FB}. It follows that IC_{IP} should bind for both the second-best allocation and the restricted second-best allocation for every $\hat{X}$; if not, there is a mixture of the second-best (restricted second-best) allocation and the first-best allocation that is incentive compatible and provides a strict welfare improvement.

Let $q_P(\hat{X}) = f_{\hat{X}} \mid [x_1(\hat{X}), x_2(\hat{X})]$, $q_I(\hat{X}) = f_{\hat{X}} \mid [0, x_1(\hat{X})] \cup [x_2(\hat{X}), x_3(\hat{X})] \cup [x_4(\hat{X}), \hat{X}]$ be the unique solution of the restricted second-best problem with parameter $\hat{X}$. Then $x_i(\hat{X}), i = 1, .., 4$ are defined uniquely unless $\hat{X} = X = \overline{X}$. Consider $\hat{X} \in [X, X']$ for some $X' > X$. By Berge's Theorem, the second-best allocation is a continuous function of $\hat{X}$. Choosing arbitrary X', we conclude that the second-best allocation is a continuous function of $\hat{X}$ for any finite $\hat{X}$. This implies that x_i, for $i = 1, ..., 4$, is also a continuous function of $\hat{X}$ for $\hat{X} \in (X, \infty)$.

The constrained optimization problem for $\hat{X} \in [X, \infty)$ with omitted IC_{IP} constraint and binding IC_{PI} constraint is then:

$$\max\left[(1-\alpha)\left(\int_0^{x_1} + \int_{x_2}^{x_3} + \int_{x_4}^{\hat{X}}\right) u_I(x)dF(x) \ + \ \alpha \int_{x_1}^{x_2} u_P(x)dF(x)\ \right]$$

$$\text{such that} \quad \begin{array}{lcll} \mu_P^{-1} \int_{x_1}^{x_2} u_P(x)dF(x) \ - \ \mu_I^{-1}\left(\int_0^{x_1} + \int_{x_2}^{x_3} + \int_{x_4}^{\hat{X}}\right) u_P(x)dF(x) & = & 0 & (\lambda) \\ F(x_1) + F(x_3) - F(x_2) + F(\hat{X}) - F(x_4) - \mu_I & = & 0 & (\rho) \\ F(x_2) - F(x_1) - \mu_P & = & 0 & (\sigma), \end{array}$$

where λ is the Lagrange multiplier associated with the incentive constraint IC_{PI}, and, therefore, $\lambda \geq 0$, and σ and ρ are Lagrange multipliers associated with the feasibility constraints. There is one degree of freedom: 3 equations for four variables. We say that a "no disposal" case occurs when $x_4 = X$ and $x_2 < x_3$, a "partial disposal" case occurs when $x_4 < X$ and $x_2 < x_3$, and a "full disposal" case occurs when $x_4 < X$ and $x_2 = x_3$. In case of no disposal or full disposal, constraints pin down the allocation uniquely. The first-order conditions are necessary and take the following form:

$$\begin{aligned}
\text{FOC}_{x_1}: &\quad (1-\alpha)u_I(x_1) - \alpha u_P(x_1) - \lambda(\mu_P^{-1} + \mu_I^{-1})u_P(x_1) + \rho - \sigma &= 0 \\
\text{FOC}_{x_2}: &\quad -(1-\alpha)u_I(x_2) + \alpha u_P(x_2) + \lambda(\mu_P^{-1} + \mu_I^{-1})u_P(x_2) - \rho + \sigma &= 0
\end{aligned}$$

$$\text{FOC}_{x_3}: \qquad (1-\alpha)u_I(x_3) - \lambda\mu_I^{-1}u_P(x_3) + \rho \qquad \begin{cases} = 0 & \text{partial disposal} \\ = 0 & \text{no disposal} \\ \leq 0 & \text{full disposal} \end{cases}$$

$$\text{FOC}_{x_4}: \qquad -(1-\alpha)u_I(x_4) + \lambda\mu_I^{-1}u_P(x_4) - \rho \qquad \begin{cases} = 0 & \text{partial disposal} \\ \geq 0 & \text{no disposal} \\ = 0 & \text{full disposal.} \end{cases}$$

The feasibility constraints allow us to express x_2 as a function of x_1, and x_4 as a function of x_3:

$$x_2(x_1) = F^{-1}\Big(\mu_P + F(x_1)\Big)\ ,\ \frac{\partial x_2}{\partial x_1} = \frac{f(x_1)}{f(x_2)} \quad , \quad x_4(x_3) = F^{-1}\Big(F(\hat{X}) + F(x_3) - \mu_P - \mu_I\Big)\ ,\ \frac{\partial x_4}{\partial x_3} = \frac{f(x_3)}{f(x_4)}.$$

The first-order conditions with respect to x_1 and x_2 allow us to express the Lagrange multiplier λ as a function of x_1 and x_2 and, therefore, as a function of x_1:

$$\lambda(x_1) \ = \ \left(\mu_P^{-1} + \mu_I^{-1}\right)^{-1} \cdot \left[(1-\alpha)\cdot\frac{u_I(x_1) - u_I(x_2(x_1))}{u_P(x_1) - u_P(x_2(x_1))} - \alpha\right].$$

Since ρ is arbitrary, the first-order conditions with respect to x_3 and x_4 are equivalent to

$$h(x_1, x_3) \ \equiv \ (1-\alpha)\Big(u_I(x_3) - u_I(x_4(x_3))\Big) - \lambda(x_1)\cdot\mu_I^{-1}\Big(u_P(x_3) - u_P(x_4(x_3))\Big) \qquad \begin{cases} = 0 & \text{partial disposal} \\ \geq 0 & \text{no disposal} \\ \leq 0 & \text{full disposal.} \end{cases}$$

In the no disposal case, $x_4 = \hat{X}$, and x_1, x_2, x_3 do not depend on $\hat{X}$. In particular, $x_3 = F^{-1}(\mu_P + \mu_I)$, $x_2 = x_2(x_1)$, and x_1 is uniquely determined by

$$(\mu_P^{-1} + \mu_I^{-1})\int_{x_1}^{x_2(x_1)} u_P(x)dF(x) \ - \ \mu_I^{-1}\int_0^{F^{-1}(\mu_P+\mu_I)} u_P(x)dF(x) = 0.$$

Consider the partial disposal case. By continuity of the allocation with respect to $\hat{X}$, the disposal remains partial in some neighborhood of $\hat{X}$ as well. Denote by

$$\phi(x_1, x_3; \hat{X}) \ \equiv \ \mu_P^{-1}\int_{x_1}^{x_2(x_1)} u_P(x)dF(x) \ - \ \mu_I^{-1}\left(\int_0^{x_1} + \int_{x_2(x_1)}^{x_3} + \int_{x_4(x_3;\hat{X})}^{\hat{X}}\right)u_P(x)dF(x).$$

Since disposal is partial, $x_3 < x_4$. Thus, $h(x_1, x_3) = 0$ is equivalent to

$$\tilde{h}(x_1, x_3) \;\equiv\; (1-\alpha)\cdot\frac{u_I(x_3)-u_I(x_4(x_3))}{u_P(x_3)-u_P(x_4(x_3))} - \mu_I^{-1}\cdot\left(\mu_P^{-1}+\mu_I^{-1}\right)^{-1}\cdot\left[(1-\alpha)\cdot\frac{u_I(x_1)-u_I(x_2(x_1))}{u_P(x_1)-u_P(x_2(x_1))}-\alpha\right] = 0.$$

Claim I1. The Jacobian of the system of equations $\tilde{h}(x_1, x_3) = 0$ and $IC(x_1, x_3; \hat{X}) = 0$ with respect to x_1 and x_3 is invertible. Moreover, $\frac{\partial \phi}{\partial x_1} < 0$, $\frac{\partial \phi}{\partial x_3} < 0$, $\frac{\partial \tilde{h}}{\partial x_1} > 0$, and $\frac{\partial \tilde{h}}{\partial x_3} < 0$.

Proof. By Lemma A2, $\frac{u_I(x_1)-u_I(x_2(x_1))}{u_P(x_1)-u_P(x_2(x_1))}$ is strictly decreasing. Thus, $\frac{\partial \tilde{h}}{\partial x_1} > 0$. Since $\frac{\partial x_4}{\partial x_3} > 0$ and $x_4 > x_3$, we can apply Lemma A2 for variables $x_1 = x_3$ and $x_2 = x_4$ to show that $\frac{u_I(x_3)-u_I(x_4(x_3))}{u_P(x_3)-u_P(x_4(x_3))}$ is strictly decreasing. Therefore, $\frac{\partial \tilde{h}}{\partial x_3} < 0$. Next,

$$\frac{\partial \phi}{\partial x_1} \;=\; -(\mu_P^{-1}+\mu_I^{-1})f(x_1)\cdot(u_P(x_1)-u_P(x_2)) \;<\; 0 \quad , \quad \frac{\partial \phi}{\partial x_3} \;=\; -\mu_I^{-1}f(x_3)\cdot(u_P(x_3)-u_P(x_4)) \;<\; 0$$

The determinant of the Jacobian matrix is $\frac{\partial \phi}{\partial x_1}\cdot\frac{\partial \tilde{h}}{\partial x_3} - \frac{\partial \phi}{\partial x_3}\cdot\frac{\partial \tilde{h}}{\partial x_1} > 0$, completing the proof. □

Since ϕ and $\tilde{h}$ are continuously differentiable functions of x_1, x_3, and $\hat{X}$, by the Implicit Function Theorem, there exists a unique differentiable solution $x_1(\hat{X}), x_3(\hat{X})$ of the system $\phi(x_1, x_3; \hat{X}) = 0$ and $\tilde{h}(x_1, x_3) = 0$. Moreover,

$$\frac{\partial \phi}{\partial \hat{X}} = \mu_I^{-1} f(\hat{X})\cdot(u_P(x_4)-u_P(\hat{X})) > 0.$$

Therefore,

$$\frac{\partial x_1}{\partial \hat{X}} \;=\; \frac{\frac{\partial \phi}{\partial x_3}\frac{\partial \tilde{h}}{\partial \hat{X}} - \frac{\partial \phi}{\partial \hat{X}}\frac{\partial \tilde{h}}{\partial x_3}}{\frac{\partial \phi}{\partial x_1}\frac{\partial \tilde{h}}{\partial x_3} - \frac{\partial \phi}{\partial x_3}\frac{\partial \tilde{h}}{\partial x_1}} \;>\; 0.$$

Thus, x_1 is a strictly increasing function of $\hat{X}$ whenever the restricted second-best allocation exhibits "partial disposal."

In the full disposal case, $x_3 = x_2(x_1)$ and the IC constraint determines x_1:

$$\phi(x_1, x_2(x_1); \hat{X}) \;=\; \mu_P^{-1}\int_{x_1}^{F^{-1}\left(\mu_P+F(x_1)\right)} u_P(x)dF(x) \;-\; \mu_I^{-1}\left(\int_0^{x_1} + \int_{F^{-1}\left(F(\hat{X})+F(x_1)-\mu_I\right)}^{\hat{X}}\right) u_P(x)dF(x),$$

$$\frac{\partial \phi(x_1, x_2(x_1); \hat{X})}{\partial x_1} \;=\; -\mu_P^{-1}\Big(u_P(x_1)-u_P(x_2)\Big)\cdot f(x_1) - \mu_I^{-1}\Big(u_P(x_1)-u_P(\hat{X})\Big)\cdot f(x_1) < 0.$$

Next,

$$\frac{\partial \phi}{\partial \hat{X}} = \mu_I^{-1} f(\hat{X})\cdot(u_P(x_4)-u_P(\hat{X})) > 0.$$

Hence,

$$\frac{\partial x_1}{\partial \hat{X}} = -\frac{\frac{\partial \phi}{\partial \hat{X}}}{\frac{\partial \phi}{\partial x_1}} > 0.$$

Since x_1 is a continuous function of $\hat{X}$, we conclude that it is increasing and, moreover, it is strictly increasing whenever there is some disposal.

We now connect the second-best allocation and the limit of the restricted second-best allocations with $\hat{X} \longrightarrow \infty$.

Claim I2. Let $x_1^{SB}, x_2^{SB}, x_3^{SB}$, and β^{SB} describe the second-best allocation following Proposition 3. Let $\hat{X}^n$ be an arbitrary sequence of parameters $\hat{X}$ such that $\hat{X}^n \longrightarrow \infty$. Let $x_i^n = x_i(\hat{X}^n)$, $i = 1, \ldots, 4$ describe the corresponding sequence of the restricted second-best allocations following Proposition 8. Then, $x_i^n \longrightarrow x_i^{SB}$ for $i = 1, 2, 3$, and $F(\hat{X}^n) - F(x_4^n) \longrightarrow \beta^{FB} \cdot \mu_I$.

Proof. Denote by q^n the restricted second-best allocation for parameter $\hat{X}^n$, and by q^{SB} the second-best allocation. By Proposition 3, we have $W(q^n) \leq W(q^{SB})$ for all n. The sequence $W(q^n)$ is weakly increasing. Let $\overline{W} = \lim_{n \to \infty} W(q^n)$.

For each $\hat{X}^n$, construct the allocation $q_P' = f \mid [x_1', x_2']$, $q_I' = f \mid [0, x_1'] \cup [x_2', x_3'] \cup [x_4', \hat{X}]$ with $x_4' = F^{-1}\Big(F(\hat{X}) - \mu_I \beta^{SB}\Big)$, $x_3' = x_3^{SB}$, $x_2' = F^{-1}\Big(F(x_1') + \mu_P\Big)$, and x_1' the unique solution of $V_P(q_P') = V_P(q_I')$. Certainly, $x_1' > x_1^{SB}$. Since $u_P(x) \longrightarrow 0$ when $x \longrightarrow \infty$, we have $V_P\Big(f \mid [0, x_1^{SB}] \cup [x_2^{SB}, x_3^{SB}] \cup [x_4', \hat{X}^n]\Big) \longrightarrow V_P(q_I^{SB}) = V_P(q_P^{SB})$. It follows that $x_i' \longrightarrow x_i^{SB}$ for $i = 1, 2, 3$, and $W(q') \longrightarrow W(q^{SB})$. Since $W(q^n) \geq W(q'(\hat{X}^n))$, then $\overline{W} = W(q^{SB})$.

For each $\hat{X}^n$, construct the allocation $q_P'' = f \mid [x_1^n, x_2^n]$, $q_I'' = (1 - \beta^n) \cdot f \mid [0, x_1^n] \cup [x_2^n, x_3^n] + \beta^n \cdot \delta_\diamond$, where $\beta^n = \mu_I^{-1} \cdot \Big(F(\hat{X}^n) - F(x_4^n)\Big)$. Since $u_P(x) \longrightarrow 0$ when $x \longrightarrow \infty$, then

$$\lim_{n \to \infty} W(q''(\hat{X}^n)) = \lim_{n \to \infty} W(q^n) = W(q^{SB}).$$

Let $\Big(x_1^{n_k}, x_2^{n_k}, x_3^{n_k}, \beta^{n_k}\Big)$ be an arbitrary subsequence of $\Big(x_1^n, x_2^n, x_3^n, \beta^n\Big)$. Since $x_i^{n_k} \in [0, X]$ for $i = 1, 2, 3$, and $\beta^n \in [0, 1]$, then there is a convergent subsubsequence $\Big(x_1^{n_{k_m}}, x_2^{n_{k_m}}, x_3^{n_{k_m}}, \beta^{n_{k_m}}\Big) \longrightarrow (x_1^*, x_2^*, x_3^*, \beta^*)$. Let q^* be the corresponding allocation: $q_P^* = f \mid [x_1^*, x_2^*]$, and $q_I^* = (1 - \beta^*) \cdot f \mid [0, x_1^*] \cup [x_2^*, x_3^*] + \beta \cdot \delta_\diamond$. Then $W(q^*) = W(q^{SB})$. Since the second-best allocation is unique, $(x_1^*, x_2^*, x_3^*, \beta^*) = (x_1^{SB}, x_2^{SB}, x_3^{SB}, \beta^{SB})$. Every subsequence of $\Big(x_1^n, x_2^n, x_3^n, \beta^n\Big)$ contains a subsubsequence converging to the same point

$(x_1^{SB}, x_2^{SB}, x_3^{SB}, \beta^{SB})$. It follows that $\left(x_1^n, x_2^n, x_3^n, \beta^n\right) \longrightarrow (x_1^{SB}, x_2^{SB}, x_3^{SB}, \beta^{SB})$, proving the claim. $\square$

Towards a contradiction, assume that there exist arbitrary large $\hat{X}$ such that the corresponding restricted second-best allocation exhibits no disposal. By Claim I2, the second-best allocation does not exhibit disposal, in contradiction. Therefore, for large enough $\hat{X}$, the restricted second-best allocation exhibits either partial or full disposal, in which case x_1 is a strictly increasing function of $\hat{X}$. We conclude that $x_1(X) < \lim_{\hat{X}\to\infty} x_1(\hat{X}) = x_1^{SB}$. It follows that $V_P(q_P(X)) > V_P(q^{SB})$. Since $W(q^{SB}) > W(q(X))$, then $V_I(q^{SB}) > W_I(q(X))$, completing the proof of the corollary. $\blacksquare$